RESEARCH NOTE

Partial sum of Dedekind sums*

ZHANG Wenpeng (张文鹏) and YI Yuan (易 媛)

(The Research Center for Basic Science, Xi'an Jiaotong University, Xi'an 710049, China)

Received October 25, 1999; revised November 24, 1999

Abstract Using the mean square value formula of Dirichlet L-functions with the weight, the distribution property of the partial sum of Dedekind sums is studied. An interesting asymptotic formula is obtained.

Keywords: Dedekind sums, partial sum, mean value distribution.

For any positive integers k and h, we define Dedekind sum $S(h, k)$ as follows:

$$S(h,k) = \sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

Because of the importance in the study of modular form theory, the Dedekind sums have attracted a lot of attention from a number of theorists. Recently, Conrey et al.[1] studied the mean value distribution of $S(h, k)$, and obtained a more general mean value theorem, i. e.

$$\sum_{h=1}^{k}{}' |S(h,k)|^{2m} = f_m(k)\left(\frac{k}{12}\right)^{2m} + O\left(\left(k^{\frac{9}{5}} + k^{2m-1+\frac{1}{m+1}}\right)\log^3 k\right), \tag{1}$$

where $\sum_h{}'$ denotes the summation over all h such that $(k, h) = 1$, and $f_m(k)$ is given by the coefficients of Dirichlet series

$$\sum_{n=1}^{\infty}\frac{f_m(n)}{n^s} = 2\frac{\zeta^2(2m)}{\zeta(4m)}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s).$$

In this paper, we study the distribution property of the partial sums of $S(h, k)$, and use the analytic method to give an interesting first power mean value theorem for it.

Theorem. *Let k be a positive integer. Then for any real number $1 < N \leqslant \frac{1}{2}k$, we have the*

* Project supported by the National Natural Science Foundation of China (Grant No. 19531010) and the Postdoctoral Science Foundation of China.

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

asymptotic formula

$$\sum_{n\leqslant N}{}' S(n,k) = \frac{1}{12}\phi(k)\left(\ln N + \gamma + \sum_{p\mid k}\frac{\ln p}{p-1}\right) + O\left(\frac{k2^{\omega(k)}}{N}\right) + O(Nk^{\varepsilon}),$$

where $\phi(k)$ *is the Euler function*, γ *the Euler constant*, $\sum_{p\mid k}$ *the summation over all different prime divisors of* k, ε *any fixed positive number and* $\omega(k)$ *the number of all different prime divisors of* k.

From this theorem we can immediately deduce the following.

Corollary. *Let* $0<\varepsilon<1$ *be any fixed positive number. Then for any real number* N *with* $k^{\varepsilon}\leqslant N\leqslant k^{1-\varepsilon}$, *the asymptotic formula*

$$\sum_{n\leqslant N}{}' S(n,k) \sim \frac{1}{12}\phi(k)\left(\ln N + \gamma + \sum_{p\mid k}\frac{\ln p}{p-1}\right)$$

holds uniformly, as $k\to+\infty$.

From the properties of Dedekind sums we know that

$$S(a,k) = -S(k-a,k) \text{ and } \sum_{a=1}^{k}{}' S(a,k) = 0.$$

On the other hand, from our theorem and its proof we can also see that if the k is large enough, then for almost all $1\leqslant a<k^{1-\varepsilon}$ we have $S(a,k)\asymp\frac{k}{12a}$. So we have many reasons to believe the following interesting conjecture.

Conjecture. *Let* k *be a positive integer and large enough, then for any positive integer* a *with* $1\leqslant a<k^{1-\varepsilon}$ *and* $(a,k)=1$, *we have*

$$S(a,k)>0.$$

1 Some lemmas

To complete the proof of the theorem, we need two elementary lemmas.

Lemma 1. *Let* $k\geqslant 3$ *be an integer with* $(h,k)=1$. *Then we have the identity*

$$S(h,k) = \frac{1}{\pi^2 k}\sum_{d\mid k}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)\left|L(1,\chi)\right|^2,$$

where χ *denotes the odd character modulo* d (*that is*, $\chi(-1)=-1$), $L(s,\chi)$ *the Dirichlet L-function corresponding to character* χ.

Proof. (See ref. [2]).

Lemma 2. *Let* k *be a positive integer, and* q *any divisor of* k *with* $q\geqslant 2$. *Then for any real number* $1<N\leqslant\frac{1}{2}k$, *we have the asymptotic formula*

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

$$\sum_{a\le N}{}' \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2 = \frac{\pi^2}{12k}\phi(k)\phi(q)\prod_{p|q}\left(1-\frac{1}{p^2}\right)\left(\ln N+\gamma+\sum_{p|k}\frac{\ln p}{p-1}\right)$$
$$+\,O\left(\frac{\phi(q)2^{\omega(k)}}{N}\prod_{p|q}\left(1-\frac{1}{p^2}\right)\right)+O(Nq^{\varepsilon}),$$

where $\sum_{a\le N}'$ *denotes the summation over all a such that* $1<a\le N$ *and* $(a,k)=1$, *and* $\prod_{p|q}$ *denotes the production over all different prime divisors of* q.

Proof. Without loss of generality, we can assume $N\le q$. Let $A(\chi,y)=\sum_{q/a<n\le y}\chi(n)$, $B(\chi,y)=\sum_{q<n\le y}\chi(n)$. Then from the Abel's identity we have

$$L(1,\chi)=\sum_{n\le q/a}\frac{\chi(n)}{n}+\int_{q/a}^{+\infty}\frac{A(\chi,y)}{y^2}\mathrm{d}y=\sum_{n\le q}\frac{\chi(n)}{n}+\int_{q}^{+\infty}\frac{B(\chi,y)}{y^2}\mathrm{d}y.$$

Hence

$$\begin{aligned}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2 &= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)\left(\sum_{n\le q/a}\frac{\chi(n)}{n}+\int_{q/a}^{+\infty}\frac{A(\chi,y)}{y^2}\mathrm{d}y\right)\\ &\quad\times\left(\sum_{n\le q}\frac{\overline{\chi}(n)}{n}+\int_{q}^{+\infty}\frac{B(\overline{\chi},y)}{y^2}\mathrm{d}y\right)\\ &= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)\left(\sum_{n\le q/a}\frac{\chi(n)}{n}\right)\left(\sum_{m\le q}\frac{\overline{\chi}(m)}{m}\right)\\ &\quad+\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)\left(\sum_{n\le q/a}\frac{\chi(n)}{n}\right)\left(\int_{q}^{+\infty}\frac{B(\overline{\chi},y)}{y^2}\mathrm{d}y\right)\\ &\quad+\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)\left(\sum_{m\le q}\frac{\overline{\chi}(m)}{m}\right)\left(\int_{q/a}^{+\infty}\frac{A(\chi,y)}{y^2}\mathrm{d}y\right)\\ &\quad+\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)\left(\int_{q/a}^{+\infty}\frac{A(\chi,y)}{y^2}\mathrm{d}y\right)\left(\int_{q}^{+\infty}\frac{B(\overline{\chi},y)}{y^2}\mathrm{d}y\right)\\ &\equiv M_1+M_2+M_3+M_4,\end{aligned}$$

Then

$$\sum_{a\le N}{}'\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2=\sum_{a\le N}{}'(M_1+M_2+M_3+M_4). \tag{2}$$

Now we estimate each term on the right hand side of (2).

(i) For $(q,mn)=1$, from the orthogonality relation for odd character we have

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(n)\overline{\chi}(m) = \begin{cases} \frac{1}{2}\phi(q), & \text{if } n \equiv m \bmod q; \\ -\frac{1}{2}\phi(q), & \text{if } n \equiv -m \bmod q; \\ 0, & \text{otherwise.} \end{cases}$$

So from this identity we obtain

$$\begin{aligned} M_1 &= \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a)\Big(\sum_{n\leqslant q/a}\frac{\chi(n)}{n}\Big)\Big(\sum_{m\leqslant q}\frac{\overline{\chi}(m)}{m}\Big) \\ &= \frac{1}{2}\phi(q)\sum_{\substack{n\leqslant q/a \\ na\equiv m(q)}}{}' \sum_{m\leqslant q}{}' \frac{1}{mn} - \frac{1}{2}\phi(q)\sum_{\substack{n\leqslant q/a \\ na\equiv -m(q)}}{}' \sum_{m\leqslant q}{}' \frac{1}{nm} \\ &= \frac{1}{2}\phi(q)\sum_{n\leqslant q/a}{}' \frac{1}{an^2} - \frac{1}{2}\phi(q)\sum_{n\leqslant q/a}{}' \frac{1}{n(q-an)} \\ &= \frac{1}{2}\phi(q)\sum_{n=1}^{\infty}{}' \frac{1}{an^2} + O\Big(\phi(q)\sum_{n>q/a}\frac{1}{an^2}\Big) + O\Big(\phi(q)\sum_{n\leqslant q/a}{}' \frac{1}{n(q-na)}\Big) \\ &= \frac{1}{2}\frac{\phi(q)}{a}\sum_{n=1}^{\infty}{}' \frac{1}{n^2} + O\Big(\frac{\phi(q)}{a}\sum_{n>q/a}\frac{1}{n^2}\Big) + O\Big(\phi(q)\sum_{n\leqslant q/2a}\frac{1}{nq}\Big) \\ &\quad + O\Big(\phi(q)\sum_{q/2a<n\leqslant \frac{q}{a}-1}\frac{a}{q(q-na)}\Big) + O\left(\frac{\phi(q)}{q}\frac{a}{q-a\left[\frac{q}{a}\right]}\right) = \\ &\quad \frac{\pi^2}{12}\frac{\phi(q)}{a}\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big) + O\Big(\frac{\phi(q)}{q}\ln q\Big) + O\left(\frac{a}{q-a\left[\frac{q}{a}\right]}\right). \end{aligned} \tag{3}$$

Then from (3) we deduce that

$$\begin{aligned} \sum_{a\leqslant N}{}' M_1 &= \sum_{a\leqslant N}{}' \left[\frac{\pi^2}{12}\frac{\phi(q)}{a}\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big) + O\Big(\frac{\phi(q)}{q}\ln q\Big) + O\left(\frac{a}{q-a\left[\frac{q}{a}\right]}\right)\right] \\ &= \frac{\pi^2}{12}\phi(q)\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)\sum_{a\leqslant N}{}' \frac{1}{a} + O\Big(\frac{N\phi(q)}{q}\ln q\Big) + O\left(\sum_{a\leqslant N}{}' \frac{a}{q-a\left[\frac{q}{a}\right]}\right). \end{aligned} \tag{4}$$

Note that

$$\sum_{a\leqslant N}{}' \frac{1}{a} = \frac{\phi(k)}{k}\Big(\ln N + \gamma + \sum_{p\mid k}\frac{\ln p}{p-1}\Big) + O\Big(\frac{2^{\omega(k)}}{N}\Big) \tag{5}$$

and

$$\sum_{a\leqslant N}{}' \frac{a}{q-a\left[\frac{q}{a}\right]} \leqslant N\sum_{u\leqslant N-1}\sum_{\substack{a\leqslant N \\ q-a\left[\frac{q}{a}\right]=u}}{}' \frac{1}{u} \leqslant N\sum_{u\leqslant N-1}\frac{d(q-u)}{u} \ll Nq^{\varepsilon}, \tag{6}$$

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

where we have used the estimate $d(n) \ll n^{\varepsilon}$ for divisor function (see Theorem 13.12 in ref. [3]).

Combining (4), (5) and (6) we may immediately get

$$\sum_{a \le N}{}' M_1 = \frac{\pi^2}{12}\phi(q)\prod_{p \mid q}\left(1 - \frac{1}{p^2}\right)\left[\frac{\phi(k)}{k}\left(\ln N + \gamma + \sum_{p \mid k}\frac{\ln p}{p-1}\right)\right]$$

$$+ O\left(\phi(q)\prod_{p \mid q}\left(1 - \frac{1}{p^2}\right)\frac{2^{\omega(k)}}{N}\right) + O(Nq^{\varepsilon})$$

$$= \frac{\pi^2}{12}\frac{\phi(k)}{k}\phi(q)\prod_{p \mid q}\left(1 - \frac{1}{p^2}\right)\left(\ln N + \gamma + \sum_{p \mid k}\frac{\ln p}{p-1}\right)$$

$$+ O\left(\phi(q)\prod_{p \mid q}\left(1 - \frac{1}{p^2}\right)\frac{2^{\omega(k)}}{N}\right) + O(Nq^{\varepsilon}). \tag{7}$$

(ii) Since $\chi(-1) = -1$ and the periodic property of the character, we can assume $B(\overline{\chi}, y) = \sum_{q < n \le y} \overline{\chi}(n) = \sum_{q < n \le y \le 2q} \overline{\chi}(n)$. So from the orthogonality relation for character sums we have

$$\sum_{\substack{\chi \bmod q \\ \chi(-1) = -1}} \chi(a)\left(\sum_{n \le q/a}\frac{\chi(n)}{n}\right)B(\overline{\chi}, y)$$

$$\ll \frac{1}{2}\phi(q)\sum_{\substack{n \le q/a}}{}'\sum_{\substack{m \le q \\ na \equiv m(q)}}{}' \frac{1}{n} + \frac{1}{2}\phi(q)\sum_{\substack{n \le q/a}}{}'\sum_{\substack{m \le q \\ na \equiv -m(q)}}{}' \frac{1}{n} \ll \phi(q)\ln q$$

and

$$\sum_{a \le N}{}' M_2 = \sum_{a \le N}{}' \int_q^{+\infty}\left[\sum_{\substack{\chi \bmod q \\ \chi(-1) = -1}} \chi(a)\left(\sum_{n \le q/a}\frac{\chi(n)}{n}\right)B(\overline{\chi}, y)\right]\frac{1}{y^2}\mathrm{d}y$$

$$\ll \sum_{a \le N}{}' \int_q^{+\infty}\frac{\phi(q)\ln q}{y^2}\mathrm{d}y \ll \frac{N}{q}\phi(q)\ln q. \tag{8}$$

(iii)

$$\sum_{a \le N}{}' M_3 = \sum_{a \le N}{}' \left[\sum_{\substack{\chi \bmod q \\ \chi(-1) = -1}} \chi(a)\left(\sum_{m \le q}\frac{\overline{\chi}(m)}{m}\right)\left(\int_{q/a}^{+\infty}\frac{A(\chi, y)}{y^2}\mathrm{d}y\right)\right]$$

$$= \sum_{a \le N}{}' \sum_{\substack{\chi \bmod q \\ \chi(-1) = -1}} \chi(a)\left(\sum_{m \le q}\frac{\overline{\chi}(m)}{m}\right)\left(\int_{q/a}^{q}\frac{A(\chi, y)}{y^2}\mathrm{d}y\right)$$

$$+ \sum_{a \le N}{}' \sum_{\substack{\chi \bmod q \\ \chi(-1) = -1}} \chi(a)\left(\sum_{m \le q}\frac{\overline{\chi}(m)}{m}\right)\left(\int_{q}^{+\infty}\frac{A(\chi, y)}{y^2}\mathrm{d}y\right)$$

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

$$= \sum_{a \leqslant N}' \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a)\left(\sum_{m \leqslant q} \frac{\overline{\chi}(m)}{m}\right)\left(\int_{q/a}^{q} \frac{\sum_{q/a<n \leqslant y} \chi(n)}{y^2} \mathrm{d}y\right) + O(N \ln q)$$

$$\ll \phi(q) \sum_{a \leqslant N}' \sum_{m \leqslant q} \frac{1}{m} \int_{q/a}^{q} \frac{1}{y^2}\Big(\sum_{\substack{q/a<n \leqslant y \\ an \equiv m(q)}} 1\Big) \mathrm{d}y + N \ln q$$

$$\ll \phi(q) \sum_{m \leqslant q} \frac{1}{m} \int_{l}^{q} \frac{y N q^{\varepsilon_1}}{q y^2} \mathrm{d}y + N \ln q \ll N q^{\varepsilon}, \tag{9}$$

where we have used the fact that for any fixed positive integers l and m, the number of the solutions of equation $an = lq + m$ (for all positive integers a and n) is $\ll q^{\varepsilon}$.

(iv) Using the same method as (iii) we can also obtain the estimation

$$\sum_{a \leqslant N}' M_4 = \sum_{a \leqslant N}' \Bigg[\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left(\int_{q/a}^{+\infty} \frac{A(\chi, y)}{y^2} \mathrm{d}y\right)\left(\int_{q}^{+\infty} \frac{B(\overline{\chi}, y)}{y^2} \mathrm{d}y\right)\Bigg] \ll N q^{\varepsilon}. \tag{10}$$

Combining (2) and (7)—(10) we may immediately get

$$\sum_{a \leqslant N}' \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a) | L(1, \chi) |^2$$

$$= \frac{\pi^2}{12} \frac{\phi(k)}{k} \phi(q) \prod_{p \mid q}\left(1 - \frac{1}{p^2}\right)\left(\ln N + \gamma + \sum_{p \mid k} \frac{\ln p}{p-1}\right)$$

$$+ O\left(\phi(q) \prod_{p \mid q}\left(1 - \frac{1}{p^2}\right) \frac{2^{\omega(k)}}{N}\right) + O(N q^{\varepsilon}).$$

This completes the proof of Lemma 2.

2 Proof of the theorem

In this section, based on Lemmas 1 and 2 we complete the proof of the theorem. In fact from Lemma 1 we can easily get the identity

$$\sum_{n \leqslant N}' S(n, k) = \sum_{n \leqslant N}' \Bigg[\frac{1}{\pi^2 k} \sum_{d \mid k} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(n) | L(1, \chi) |^2\Bigg]$$

$$= \frac{1}{\pi^2 k} \sum_{d \mid k} \frac{d^2}{\phi(d)} \sum_{n \leqslant N}' \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(n) | L(1, \chi) |^2. \tag{11}$$

Using the results of Lemma 2 in (11) we have

$$\sum_{n \leqslant N}' S(n, k) = \frac{\phi(k)}{12 k^2} \sum_{d \mid k} d^2 \prod_{p \mid d}\left(1 - \frac{1}{p^2}\right)\left(\ln N + \gamma + \sum_{p \mid k} \frac{\ln p}{p-1}\right)$$

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

$$+\frac{1}{\pi^2 k}\sum_{d\mid k}\frac{d^2}{\phi(d)}\left[O\left(\phi(d)\prod_{p\mid d}\left(1-\frac{1}{p^2}\right)\frac{2^{\omega(k)}}{N}\right)+O(Nd^{\varepsilon})\right]$$

$$=\frac{1}{12}\phi(k)\left(\ln N+\gamma+\sum_{p\mid k}\frac{\ln p}{p-1}\right)+O\left(\frac{k2^{\omega(k)}}{N}\right)+O(Nk^{\varepsilon}),$$

where we have used the identity

$$\sum_{d\mid k}d^2\prod_{p\mid d}\left(1-\frac{1}{p^2}\right)=k^2.$$

This completes the proof of the theorem.

Acknowledgements The authors express their gratitude to the referees for their very helpful and detailed comments.

References

1 Conrey, J. B., Fransen, E., Klein, R. et al., Mean values of Dedekind sums, *Journal of Number Theory*, 1996, 56: 214.

2 Zhang, W. P., On the mean values of Dedekind sums, *Journal de Theorie des Nombres de Bordeaux*, 1996, 8: 429.

3 Apostol T. M., *Introduction to Analytic Number Theory*, New York: Springer-Verlag, 1976.

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net